

YANG-BAXTER SYSTEMS AND ENTWINING STRUCTURES

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ABSTRACT. It is shown that a Yang-Baxter system can be constructed from any entwining structure. It is also shown that, conversely, Yang-Baxter systems of certain type lead to entwining structures. Examples of Yang-Baxter systems associated to entwining structures are given, and a Yang-Baxter operator of Hecke type is defined for any bijective entwining map.

1. INTRODUCTION AND PRELIMINARIES

The origins of the quantum Yang-Baxter equation are in theoretical physics, where it first appeared in the context of factorisation of S -matrices and in the quantum inverse scattering method of solving integrable Hamiltonian systems. Since the work of Drinfeld and Jimbo, Yang-Baxter equations are recognised as the origin of the theory of quantum groups or quasitriangular Hopf algebras. They are the starting point of many developments in knot theory and in the theory of braided monoidal categories. Due to the complexity of various integrable models the need arose for extensions and generalisations of Yang-Baxter equations and related algebraic structures. One such generalisation, termed a *Yang-Baxter system*, was proposed in [10] in the context of non-ultralocal integrable systems [9].

Motivated by the need for developing a general theory of non-commutative principal bundles, entwining structures were introduced in [4] as generalised symmetries of such bundles. In recent years, entwining structures and categories of modules associated to them have been used to provide a unification of various types of Hopf modules (cf. [1], [6]), and eventually have led to the revival of the theory of corings (cf. [2], [5]).

Thus Yang-Baxter systems and entwining structures arose in entirely different contexts, were motivated by as far fields as high energy physics and non-commutative geometry, and have entirely different applications. There is no reason to expect that there is any connection between these two notions. Yet, in this paper, we prove that such a connection, and a very close one at that, indeed exists. More precisely, we show that to any entwining structure one can associate a Yang-Baxter system. The converse is also true: to any Yang-Baxter system of certain type one can associate an entwining structure.

The remainder of this section contains preliminaries about Yang-Baxter operators. The main results and examples are contained in Section 2. We first recall the notions of a Yang-Baxter system and an entwining structure and then prove the theorem connecting these two notions. We then present a series of examples of Yang-Baxter systems related to various entwining structures. We also use the classification of two-dimensional Yang-Baxter systems given in [10] to construct an explicit example of an entwining structure. Finally, we show that to every entwining structure $(A, C)_\psi$ with a bijective entwining

map ψ , one can associate a q -Hecke operator on $A \oplus C$ obtained by gluing q -Hecke operators on A and C .

Throughout this paper k is a field. The unadorned tensor product is over k . The identity map on a k -vector space V is denoted by I_V or simply by I provided the domain is clear from the context. All algebras are over k , they are associative and with unit 1. The product in an algebra A is denoted by $\mu : A \otimes A \rightarrow A$, while the unit map is denoted by $\iota : k \rightarrow A$. Similarly, all coalgebras are over k , they are coassociative and with a counit. Coproduct in a coalgebra C is denoted by $\Delta : C \rightarrow C \otimes C$ and the counit by $\epsilon : C \rightarrow k$. We use the standard notation for coalgebras and comodules. In particular, for a coalgebra C , we use Sweedler's notation to denote the coproduct Δ on elements, i.e., $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$, for all $c \in C$. For a right C -comodule V , the coaction is denoted by $\varrho^V(v) = \sum v_{(0)} \otimes v_{(1)}$, for all $v \in V$.

For any vector spaces V and W , $\tau_{V,W} : V \otimes W \rightarrow W \otimes V$ denotes the natural bijection defined by $\tau_{V,W}(v \otimes w) = w \otimes v$. Let $R : V \otimes V \rightarrow V \otimes V$ be a k -linear map. Define $R_{12} = R \otimes I_V$, $R_{23} = I_V \otimes R$ and $R_{13} = (I_V \otimes \tau_{V,V}) \circ (R \otimes I_V) \circ (I_V \otimes \tau_{V,V})$. Each of the R_{ij} is thus a linear endomorphism of $V \otimes V \otimes V$. Recall that an invertible k -linear map $R : V \otimes V \rightarrow V \otimes V$ is called a *Yang-Baxter operator* (or simply a *YB operator*) if it satisfies the equation

$$(1.1) \quad R_{12} \circ R_{23} \circ R_{12} = R_{23} \circ R_{12} \circ R_{23}.$$

The equation (1.1) is usually called the *braid equation*. It is a well-known fact that the operator R satisfies (1.1) if and only if $R \circ \tau_{V,V}$ satisfies the *quantum Yang-Baxter equation*

$$(1.2) \quad R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12},$$

perhaps more familiar in physics and quantum group theory. For a review of Yang-Baxter operators we refer to [11].

In what follows, we use the following construction of Yang-Baxter operators described in [7]. If A is a k -algebra, then for all non-zero $r, s \in k$, the linear map

$$(1.3) \quad R_{r,s}^A : A \otimes A \rightarrow A \otimes A, \quad a \otimes b \mapsto sab \otimes 1 + r1 \otimes ab - sa \otimes b$$

is a Yang-Baxter operator. The inverse of $R_{r,s}^A$ is $(R_{r,s}^A)^{-1}(a \otimes b) = \frac{1}{r}ab \otimes 1 + \frac{1}{s}1 \otimes ab - \frac{1}{s}a \otimes b$. Dually, if C is a coalgebra, then for all non-zero $p, t \in k$, the linear map

$$(1.4) \quad R_C^{p,t} : C \otimes C \rightarrow C \otimes C, \quad c \otimes d \mapsto p\epsilon(c) \sum d_{(1)} \otimes d_{(2)} + t\epsilon(d) \sum c_{(1)} \otimes c_{(2)} - pc \otimes d$$

is a Yang-Baxter operator. Note that in both cases the assumption that parameters are non-zero is needed only for the invertibility of $R_{r,s}^A$ and $R_C^{p,t}$. $R_{r,s}^A$ and $R_C^{p,t}$ satisfy the braid relation for any value of r, s, p and t .

2. FROM ENTWINING STRUCTURES TO YANG-BAXTER SYSTEMS (AND BACK)

Yang-Baxter systems were introduced in [10] as a spectral-parameter independent generalisation of quantum Yang-Baxter equations related to non-ultralocal integrable systems studied previously in [9]. Yang-Baxter systems are conveniently defined in terms of *Yang-Baxter commutators*. Consider three vector spaces V, V', V'' and three linear maps $R : V \otimes V' \rightarrow V \otimes V'$, $S : V \otimes V'' \rightarrow V \otimes V''$ and $T : V' \otimes V'' \rightarrow V' \otimes V''$. Then a *Yang-Baxter commutator* is a map $[R, S, T] : V \otimes V' \otimes V'' \rightarrow V \otimes V' \otimes V''$, defined by

$$(2.5) \quad [R, S, T] = R_{12} \circ S_{13} \circ T_{23} - T_{23} \circ S_{13} \circ R_{12}.$$

In terms of a Yang-Baxter commutator, the quantum Yang-Baxter equation (1.2) is expressed simply as $[R, R, R] = 0$.

Definition 2.1. Let V and V' be vector spaces. A system of linear maps

$$W : V \otimes V \rightarrow V \otimes V, \quad Z : V' \otimes V' \rightarrow V' \otimes V', \quad X : V \otimes V' \rightarrow V \otimes V'$$

is called a *WXZ-system* or a *Yang-Baxter system*, provided the following equations are satisfied:

$$(2.6) \quad [W, W, W] = 0,$$

$$(2.7) \quad [Z, Z, Z] = 0,$$

$$(2.8) \quad [W, X, X] = 0,$$

$$(2.9) \quad [X, X, Z] = 0.$$

There are several algebraic origins and applications of WXZ-systems. It has been observed in [16] that WXZ-systems with invertible W , X and Z can be used to construct dually-paired bialgebras of the FRT type, thus leading to quantum doubles. More precisely, consider a WXZ-system with finite-dimensional $V = V'$, so that each of W , X , Z is an $N^2 \times N^2$ -matrix. Suppose that W, X, Z are invertible. Since W and Z satisfy Yang-Baxter equations (2.6)–(2.7), one can consider two matrix bialgebras A and B with $N \times N$ matrices of generators U and T respectively, and relations $W_{12}U_1U_2 = U_2U_1W_{12}$, $Z_{12}T_1T_2 = T_2T_1Z_{12}$ (cf. [8]). The existence of an invertible operator X that satisfies equations (2.8)–(2.9), means that A and B are dually paired with a non-degenerate pairing $\langle U_1, T_2 \rangle = X_{12}$. Furthermore, the tensor product $A \otimes B$ has an algebra (quantum double) structure with crossed relations $X_{12}U_1T_2 = T_2U_1X_{12}$.

Given a WXZ-system as in Definition 2.1 one can construct a Yang-Baxter operator on $V \oplus V'$, provided the map X is invertible. This is a special case of a *gluing procedure* described in [13, Theorem 2.7] (cf. [13, Example 2.11]). Let $R = W \circ \tau_{V,V}$, $R' = Z \circ \tau_{V',V'}$, $U = X \circ \tau_{V',V}$. Then the linear map

$$R \oplus_U R' : (V \oplus V') \otimes (V \oplus V') \rightarrow (V \oplus V') \otimes (V \oplus V')$$

given by $R \oplus_U R'|_{V \otimes V} = R$, $R \oplus_U R'|_{V' \otimes V'} = R'$, and for all $x \in V$, $y \in V'$,

$$(R \oplus_U R')(y \otimes x) = U(y \otimes x), \quad (R \oplus_U R')(x \otimes y) = U^{-1}(x \otimes y)$$

is a Yang-Baxter operator.

Entwining structures were introduced in [4] in order to recapture the symmetry structure of non-commutative (coalgebra) principal bundles or coalgebra-Galois extensions. For applications and algebraic content of entwining structures we refer to [6] and [5].

Definition 2.2. An algebra A is said to be *entwined* with a coalgebra C if there exists a linear map $\psi : C \otimes A \rightarrow A \otimes C$ satisfying the following four conditions:

- (1) $\psi \circ (I_C \otimes \mu) = (\mu \otimes I_C) \circ (I_A \otimes \psi) \circ (\psi \otimes I_A)$,
- (2) $(I_A \otimes \Delta) \circ \psi = (\psi \otimes I_C) \circ (I_C \otimes \psi) \circ (\Delta \otimes I_A)$,
- (3) $\psi \circ (I_C \otimes \iota) = \iota \otimes I_C$,

$$(4) (I_A \otimes \epsilon) \circ \psi = \epsilon \otimes I_A.$$

The map ψ is known as an *entwining map*, and the triple $(A, C)_\psi$ is called an *entwining structure*.

To denote the action of an entwining map ψ on elements it is convenient to use the following α -notation, for all $a, b \in A$ and $c \in C$,

$$\psi(c \otimes a) = \sum_{\alpha} a_{\alpha} \otimes c^{\alpha}, \quad (I_A \otimes \psi) \circ (\psi \otimes I_A)(c \otimes a \otimes b) = \sum_{\alpha, \beta} a_{\alpha} \otimes b_{\beta} \otimes c^{\alpha\beta},$$

etc. The relations (1), (2), (3) and (4) in Definition 2.2 are equivalent to the following explicit relations, for all $a, b \in A$, $c \in C$,

$$(2.10) \quad \sum_{\alpha} (ab)_{\alpha} \otimes c^{\alpha} = \sum_{\alpha, \beta} a_{\alpha} b_{\beta} \otimes c^{\alpha\beta}$$

$$(2.11) \quad \sum_{\alpha} a_{\alpha} \otimes c^{\alpha}_{(1)} \otimes c^{\alpha}_{(2)} = \sum_{\alpha, \beta} a_{\beta\alpha} \otimes c_{(1)}^{\alpha} \otimes c_{(2)}^{\beta}$$

$$(2.12) \quad \sum_{\alpha} 1_{\alpha} \otimes c^{\alpha} = 1 \otimes c$$

$$(2.13) \quad \sum_{\alpha} a_{\alpha} \epsilon(c^{\alpha}) = a \epsilon(c)$$

The main result of this paper is contained in the following

Theorem 2.3. *Let A be an algebra and let C be a coalgebra. For any $s, r, t, p \in k$ define linear maps*

$$W : A \otimes A \rightarrow A \otimes A, \quad a \otimes b \mapsto sba \otimes 1 + r1 \otimes ba - sb \otimes a,$$

$$Z : C \otimes C \rightarrow C \otimes C, \quad c \otimes d \mapsto t\epsilon(c) \sum d_{(1)} \otimes d_{(2)} + p\epsilon(d) \sum c_{(1)} \otimes c_{(2)} - pd \otimes c.$$

Let $X : A \otimes C \rightarrow A \otimes C$ be a linear map such that $X \circ (\iota \otimes I_C) = \iota \otimes I_C$ and $(I_A \otimes \epsilon) \circ X = I_A \otimes \epsilon$. Then W, X, Z is a Yang-Baxter system if and only if A is entwined with C by the map $\psi := X \circ \tau_{C,A}$.

Proof. Note that $W = R_{r,s}^A \circ \tau_{A,A}$ and $Z = R_C^{p,t} \circ \tau_{C,C}$, where $R_{r,s}^A$ is defined in (1.3), and $R_C^{p,t}$ is defined in (1.4), so that the conditions $[W, W, W] = 0$ and $[Z, Z, Z] = 0$ are satisfied.

Suppose that $(A, C)_\psi$ is an entwining structure. Equations (2.12)–(2.13) imply that $X \circ (\iota \otimes I_C) = \iota \otimes I_C$ and $(I_A \otimes \epsilon) \circ X = I_A \otimes \epsilon$. The fact that $X = \psi \circ \tau_{A,C}$ satisfies conditions (2.8)–(2.9) can be checked by direct computations. Explicitly, using the α -notation we obtain $X(a \otimes c) = \sum_{\alpha} a_{\alpha} \otimes c^{\alpha}$, for all $a \in A$ and $c \in C$. In view of the definition of W , $W_{12}(a \otimes b \otimes c) = sba \otimes 1 \otimes c + r1 \otimes ba \otimes c - sb \otimes a \otimes c$, for all $a, b \in A$ and $c \in C$, so that

$$X_{13} \circ W_{12}(a \otimes b \otimes c) = s \sum_{\alpha} (ba)_{\alpha} \otimes 1 \otimes c^{\alpha} + r1 \otimes ba \otimes c - s \sum_{\alpha} b_{\alpha} \otimes a \otimes c^{\alpha}.$$

The second expression on the right-hand side is obtained with the help of formula (2.12). Again the use of (2.12) leads to

$$X_{23} \circ X_{13} \circ W_{12}(a \otimes b \otimes c) = s \sum_{\alpha} (ba)_{\alpha} \otimes 1 \otimes c^{\alpha} + r \sum_{\alpha} 1 \otimes (ba)_{\alpha} \otimes c^{\alpha} - s \sum_{\alpha, \beta} b_{\alpha} \otimes a_{\beta} \otimes c^{\alpha\beta}.$$

On the other hand, the definition of X yields $X_{23}(a \otimes b \otimes c) = \sum_{\alpha} a \otimes b_{\alpha} \otimes c^{\alpha}$, so that $X_{13} \circ X_{23}(a \otimes b \otimes c) = \sum_{\alpha, \beta} a_{\beta} \otimes b_{\alpha} \otimes c^{\alpha\beta}$, and therefore

$$W_{12} \circ X_{13} \circ X_{23}(a \otimes b \otimes c) = s \sum_{\alpha, \beta} b_{\alpha} a_{\beta} \otimes 1 \otimes c^{\alpha\beta} + r \sum_{\alpha, \beta} 1 \otimes b_{\alpha} a_{\beta} \otimes c^{\alpha\beta} - s \sum_{\alpha, \beta} b_{\alpha} \otimes a_{\beta} \otimes c^{\alpha\beta}.$$

Now, using equation (2.10) for the first and second terms on the right-hand side, we obtain $[W, X, X] = 0$, as required. Thus (2.8) holds.

The equality $[X, X, Z] = 0$ can be deduced from the self-duality of an entwining structure. More precisely, the conditions Definition 2.2 are invariant under the operation consisting of exchanging A with C , μ with Δ , ι with ϵ and reversing the order of the composition (no change in ψ). As a result of this operation, condition (1) in Definition 2.2 becomes condition (2), condition (2) becomes condition (1), and (3) is exchanged with (4) - but there is no overall change in the content of Definition 2.2. The above operation changes W to a map of the form Z , and the Yang-Baxter commutator $[W, X, X]$ to the commutator of the form $[X, X, Z]$. Since the calculation of $[W, X, X]$ does not depend on the choice of the parameters r, s , $[W, X, X] = 0$ implies $[X, X, Z] = 0$. This can also be checked explicitly. Directly from the definitions of X and Z one finds, for all $a \in A$ and $c, d \in C$,

$$\begin{aligned} X_{12} \circ X_{13} \circ Z_{23}(a \otimes c \otimes d) &= t\epsilon(c) \sum_{\beta, \alpha} a_{\beta\alpha} \otimes d_{(1)}^{\alpha} \otimes d_{(2)}^{\beta} \\ &\quad + p\epsilon(d) \sum_{\beta, \alpha} a_{\beta\alpha} \otimes c_{(1)}^{\alpha} \otimes c_{(2)}^{\beta} - p \sum_{\beta, \alpha} a_{\beta\alpha} \otimes d^{\alpha} \otimes c^{\beta} \end{aligned}$$

and

$$\begin{aligned} Z_{23} \circ X_{13} \circ X_{12}(a \otimes c \otimes d) &= t \sum_{\alpha, \beta} \epsilon(c^{\alpha}) a_{\alpha\beta} \otimes d^{\beta}_{(1)} \otimes d^{\beta}_{(2)} \\ &\quad + p \sum_{\alpha, \beta} a_{\alpha\beta} \epsilon(d^{\beta}) \otimes c^{\alpha}_{(1)} \otimes c^{\alpha}_{(2)} - p \sum_{\alpha, \beta} a_{\alpha\beta} \otimes d^{\beta} \otimes c^{\alpha}. \end{aligned}$$

Next, apply equation (2.13) (dual to equation (2.12)) to the first two terms on the right-hand side of the last equality to obtain

$$\begin{aligned} Z_{23} \circ X_{13} \circ X_{12}(a \otimes c \otimes d) &= t \sum_{\alpha, \beta} \epsilon(c) a_{\beta} \otimes d^{\beta}_{(1)} \otimes d^{\beta}_{(2)} \\ &\quad + p \sum_{\alpha} a_{\alpha} \epsilon(d) \otimes c^{\alpha}_{(1)} \otimes c^{\alpha}_{(2)} - p \sum_{\alpha, \beta} a_{\alpha\beta} \otimes d^{\beta} \otimes c^{\alpha}. \end{aligned}$$

Finally, application of equation (2.11) (dual to equation (2.10)) to the first two terms on the right-hand side yields $[X, X, Z] = 0$, as expected.

In the converse direction, suppose that the map $X : A \otimes C \rightarrow A \otimes C$ is such that W , X and Z form a Yang-Baxter system and that $X \circ (\iota \otimes I_C) = \iota \otimes I_C$ and $(I_A \otimes \epsilon) \circ X = I_A \otimes \epsilon$. The latter immediately imply that $\psi = X \circ \tau_{C,A}$ satisfies conditions (3) and (4) in Definition 2.2. Write $\psi(c \otimes a) = \sum_{\alpha} a_{\alpha} \otimes c^{\alpha}$, and perform the same computations of $[W, X, X]$ and $[X, X, Z]$ as before (note that one can use equations (2.12)–(2.13)), to conclude that $[W, X, X] = 0$ imply that, for all $a, b \in A$, $c \in C$,

$$\sum_{\alpha} 1 \otimes (ba)_{\alpha} \otimes c^{\alpha} = \sum_{\alpha, \beta} 1 \otimes b_{\alpha} a_{\beta} \otimes c^{\alpha\beta},$$

so that equation (2.10) holds. Furthermore, the equality $[X, X, Z] = 0$ implies that, for all $a \in A$, $c, d \in C$,

$$\sum_{\alpha} a_{\alpha} \epsilon(d) \otimes c^{\alpha}_{(1)} \otimes c^{\alpha}_{(2)} = \epsilon(d) \sum_{\beta, \alpha} a_{\beta\alpha} \otimes c_{(1)}^{\alpha} \otimes c_{(2)}^{\beta}.$$

It therefore follows that for all $a \in A$, $c, d \in C$,

$$d \otimes \sum_{\alpha} a_{\alpha} \otimes c^{\alpha}_{(1)} \otimes c^{\alpha}_{(2)} = d \otimes \sum_{\beta, \alpha} a_{\beta\alpha} \otimes c_{(1)}^{\alpha} \otimes c_{(2)}^{\beta}.$$

Since k is a field, the above equality yields equation (2.11). Thus ψ is an entwining map. \square

Remark 2.4. As observed in [4, Proposition 2.7], entwining structures are related to *algebra factorisations* via semi-dualisation (see [12, p. 300] for discussion of algebra factorisations). The arguments similar to those in the proof of Theorem 2.3 show that, given two algebras A , B , and an algebra factorisation map $\Psi : B \otimes A \rightarrow A \otimes B$, one can construct a Yang-Baxter system with W and Z of the same form as W in Theorem 2.3 and $X = \Psi \circ \tau_{A,B}$.

Thus Theorem 2.3 implies that for any entwining structure $(A, C)_{\psi}$ there is a family (parametrised by four elements of k) of Yang-Baxter systems. We describe two general examples below.

Example 2.5. Yang-Baxter systems associated to a Doi-Koppinen datum.

Let B be a bialgebra, A be a right B -comodule algebra and let C be a right B -module coalgebra. Then, A is entwined with C via $\psi : C \otimes A \rightarrow A \otimes C$, $\psi(c \otimes a) = \sum a_{(0)} \otimes ca_{(1)}$ (cf. [1, Example 3.1]). Therefore, for all $r, s, p, t \in k$, the maps

$$W : A \otimes A \rightarrow A \otimes A, \quad a \otimes b \mapsto sba \otimes 1 + r1 \otimes ba - sb \otimes a,$$

$$X : A \otimes C \rightarrow A \otimes C, \quad a \otimes c \mapsto \sum a_{(0)} \otimes ca_{(1)},$$

$$Z : C \otimes C \rightarrow C \otimes C, \quad c \otimes d \mapsto t\epsilon(c) \sum d_{(1)} \otimes d_{(2)} + p\epsilon(d) \sum c_{(1)} \otimes c_{(2)} - pd \otimes c$$

form a Yang-Baxter system. Furthermore, if B^{op} is a Hopf algebra (i.e., there exists a map $S : B \rightarrow B$ such that $\sum S(b_{(2)})b_{(1)} = \sum b_{(2)}S(b_{(1)}) = \epsilon(b)$) then ψ is invertible and thus so is X .

As a particular case we can take $A = C = B$ and view B as a B -comodule algebra via Δ and as a B -module coalgebra via μ . Thus any bialgebra gives rise to a Yang-Baxter system.

Example 2.6. Yang-Baxter systems associated to coalgebra-Galois extensions. Let C be a coalgebra, and A be an algebra and a right C -comodule with a coaction $\rho^A : A \rightarrow A \otimes C$. Define a subalgebra B of A by

$$B = \{b \in A \mid \forall a \in A, \rho^A(ba) = b\rho^A(a) = \sum ba_{(0)} \otimes a_{(1)}\}.$$

Suppose that the canonical right C -comodule, left B -module map

$$\text{can} : A \otimes_B A \rightarrow A \otimes C, \quad a \otimes a' \mapsto a\rho^A(a')$$

is bijective, i.e., that $B \subseteq A$ is a *coalgebra-Galois C -extension*. Then A is entwined with C via the map $\psi : C \otimes A \rightarrow A \otimes C$, $c \otimes a \mapsto \text{can}(\text{can}^{-1}(1 \otimes c)a)$ (cf. [3, Theorem 2.7]). Hence, for all $r, s, p, t \in k$, the maps

$$\begin{aligned} W : A \otimes A &\rightarrow A \otimes A, & a \otimes b &\mapsto sba \otimes 1 + r1 \otimes ba - sb \otimes a, \\ X : A \otimes C &\rightarrow A \otimes C, & a \otimes c &\mapsto \text{can}(\text{can}^{-1}(1 \otimes c)a), \\ Z : C \otimes C &\rightarrow C \otimes C, & c \otimes d &\mapsto t\epsilon(c) \sum d_{(1)} \otimes d_{(2)} + p\epsilon(d) \sum c_{(1)} \otimes c_{(2)} - pd \otimes c \end{aligned}$$

form a Yang-Baxter system.

The following example is based on the construction of an entwining structure in [15, Example 3.4]

Example 2.7. Let C be a coalgebra spanned as a vector space by e and y_j , $j \in \mathbb{Z}$ with the coproduct $\Delta(e) = e \otimes e$, $\Delta(y_j) = e \otimes y_j + y_j \otimes e$ and the counit $\epsilon(e) = 1$, $\epsilon(y_j) = 0$. Let $A = k[x_i \mid i \in \mathbb{Z}]/(x_i x_j \mid i, j \in \mathbb{Z})$. Define a linear map $\psi : C \otimes A \rightarrow A \otimes C$ by

$$\psi(c \otimes 1) = 1 \otimes c, \quad \psi(e \otimes a) = a \otimes e, \quad \psi(y_j \otimes x_i) = x_{i+1} \otimes y_{j+1},$$

for all $a \in A$, $c \in C$. Then $(A, C)_\psi$ is an entwining structure, and hence, for all $r, s, p, t \in k$, $W : A \otimes A \rightarrow A \otimes A$, $X : A \otimes C \rightarrow A \otimes C$, $Z : C \otimes C \rightarrow C \otimes C$, defined by,

$$\begin{aligned} W(1 \otimes 1) &= r1 \otimes 1, & W(x_i \otimes 1) &= sx_i \otimes 1 + (r - s)1 \otimes x_i, \\ W(1 \otimes x_i) &= r1 \otimes x_i, & W(x_i \otimes x_j) &= -sx_j \otimes x_i, \\ X(1 \otimes e) &= 1 \otimes e, & X(1 \otimes y_i) &= 1 \otimes y_i, & X(x_i \otimes e) &= x_i \otimes e, & X(x_i \otimes y_j) &= x_{i+1} \otimes y_{j+1} \\ Z(e \otimes e) &= te \otimes e, & Z(e \otimes y_j) &= te \otimes y_j + (t - p)y_j \otimes e, \\ Z(y_j \otimes e) &= py_j \otimes e, & Z(y_i \otimes y_j) &= -py_j \otimes y_i, \end{aligned}$$

form a Yang-Baxter system.

The final example we consider will illustrate how, in view of Theorem 2.3 the knowledge of Yang-Baxter systems can lead to construction (and classification) of entwining structures with invertible entwining map. We use the full classification of WXZ-systems in dimension two, i.e., $\dim_k V = \dim_k V' = 2$, obtained in [10].

Example 2.8. Guided by the classification of WXZ-systems in [10], for any $s \in k$ ($s \neq -1$), consider a two-dimensional algebra A with basis $\{1, x\}$ and with relation $x^2 = \frac{1}{s+1}$, and a two-dimensional coalgebra C with basis $\{e, f\}$ and the coproduct and counit

$$\Delta(e) = e \otimes e + \frac{1}{s+1} f \otimes f, \quad \Delta(f) = e \otimes f + f \otimes e, \quad \epsilon(e) = 1, \quad \epsilon(f) = 0.$$

Take W and Z as in Theorem 2.3 with $r = t = 1$ and $p = s$. Thus explicitly, the operator W reads

$$W(1 \otimes 1) = 1 \otimes 1, \quad W(1 \otimes x) = 1 \otimes x,$$

$$W(x \otimes 1) = (1 - s)1 \otimes x + sx \otimes 1, \quad W(x \otimes x) = 1 \otimes 1 - sx \otimes x,$$

or in the matrix form

$$(2.14) \quad W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 - s & s & 0 \\ 1 & 0 & 0 & -s \end{pmatrix},$$

and Z is described by a matrix that is the transpose of matrix W .

With these operators at hand we look at the list of solutions for Yang-Baxter systems in [10], to find out that the only relevant cases, i.e., those in which W and Z come from algebras and coalgebras, are the cases 1, 2, 56 and 59 in there. In the case 1, the entwining map comes out as an ordinary twist. Furthermore, 2 is a special case of 56, the latter described by the matrix

$$(2.15) \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The corresponding entwining map $\psi : C \otimes A \rightarrow A \otimes C$ comes out as

$$(2.16) \quad \psi(e \otimes x) = q1 \otimes f + x \otimes e, \quad \psi(f \otimes x) = -x \otimes f.$$

Finally the case 59 is described by the matrix

$$(2.17) \quad X = \begin{pmatrix} s^{-1} & 0 & 0 & q(1 - s) \\ 0 & 1 & 0 & 0 \\ 0 & q & s^{-1} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This means in particular that $X(1 \otimes e) = s^{-1}1 \otimes e + q(1 - s)x \otimes f$. Thus to satisfy assumptions of Theorem 2.3 one has to set $s = 1$, in which case the corresponding entwining map coincides with the map given by equations (2.16).

We have already explained a connection between Yang-Baxter systems and Yang-Baxter operators obtained by the gluing procedure described in [13]. Given an entwining structure $(A, C)_\psi$ such that ψ is bijective and provided that an appropriate choice of the parameters in Theorem 2.3 is made, in addition to the Yang-Baxter operator on $A \oplus C$ discussed earlier, one can construct another Yang-Baxter operator R such that $(R + q^{-1}I) \circ (R - qI) = 0$ for a non-zero $q \in k$. Such a Yang-Baxter operator is known as a *q-Hecke operator*. This follows from [13, Example 2.11] and can be stated as the following

Proposition 2.9. *Let $(A, C)_\psi$ be an entwining structure and let W and Z be as in Theorem 2.3. Suppose that $r = t = q$ and $s = p = q^{-1}$, for a non-zero $q \in k$, and that ψ is bijective. Let $R = W \circ \tau_{A,A}$ and $R' = Z \circ \tau_{C,C}$ and define a linear map*

$$R \oplus_\psi R' : (A \oplus C) \otimes (A \oplus C) \rightarrow (A \oplus C) \otimes (A \oplus C)$$

by $R \oplus_\psi R'|_{A \otimes A} = R$, $R \oplus_\psi R'|_{C \otimes C} = R'$, and, for all $a \in A$, $c \in C$,

$$(R \oplus_\psi R')(a \otimes c) = \psi^{-1}(a \otimes c) \quad (R \oplus_\psi R')(c \otimes a) = \psi(c \otimes a) + (q - q^{-1})c \otimes a.$$

Then $R \oplus_\psi R'$ is a *q-Hecke operator*.

Proof. Note that $R = R_{q,q^{-1}}^A$, where $R_{r,s}^A$ is given in equation (1.3), while $R' = R_C^{q^{-1},q}$, where $R_C^{p,t}$ is given in (1.4). The minimal polynomials of $R_{r,s}^A$ and $R_C^{p,t}$ are $(R_{r,s}^A + s)(R_{r,s}^A - r)$ and $(R_C^{p,t} + p)(R_C^{p,t} - t)$, so that both $R : A \otimes A \rightarrow A \otimes A$ and $R' : C \otimes C \rightarrow C \otimes C$ are q -Hecke operators. In view of Theorem 2.3, [13, Example 2.11] implies that also $R \oplus_\psi R'$ is a q -Hecke operator. \square

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